THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE two hundred and twelfth regular meeting of the Society was held at Columbia University on Saturday, October 30, 1920, extending through the usual morning and afternoon sessions. The attendance included the following thirty-five members of the Society:

Dr. J. W. Alexander, Professor A. A. Bennett, Professor F. N. Cole, Dr. Tobias Dantzig, Dr. Jesse Douglas, Professor L. P. Eisenhart, Professor H. B. Fine, Professor T. S. Fiske, Professor W. B. Fite, Mr. Philip Franklin, Dr. T. H. Gronwall, Dr. C. M. Hebbert, Dr. A. A. Himwich, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. E. A. T. Kircher, Dr. K. W. Lamson, Mr. Harry Langman, Dr. H. F. MacNeish, Professor Frank Morley, Professor W. F. Osgood, Dr. G. A. Pfeiffer, Dr. E. L. Post, Professor H. W. Reddick, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor H. W. Tyler, Professor Oswald Veblen, Professor J. H. M. Wedderburn, Mr. R. A. Wetzel, Professor H. S. White, Professor J. K. Whittemore, Dr. T. S. Yang.

President Morley occupied the chair. The Council announced the election of the following persons to membership in the Society: Dr. P. M. Batchelder, University of Texas; Miss Vevia Blair, Horace Mann School; Mr. E. H. Carus, La Salle, Ill.; Mr. W. E. Cederberg, University of Wisconsin; Mr. R. P. Conkling, Newark Technical School; Mr. P. H. Evans, Northwestern Mutual Life Insurance Company, Milwaukee, Wis.; Mr. B. L. Falconer, U. S. Civil Service Commission, Boston, Mass.; Mr. J. A. Foberg, Crane Junior College, Chicago, Ill.; Dr. Gladys E. C. Gibbens, University of Minnesota; Professor L. E. Gurney, University of the Philippines; Professor Archibald Henderson, University of North Carolina; Miss Jewell C. Hughes, University of Arkansas; Miss Claribel Kendall, University of Colorado; Mrs. M. I. Logsdon, University of Chicago; Mr. R. L. McNeal, General Motors Laboratories, Detroit, Mich.; Mr. H. L. Olson, University of Michigan; Professor Leigh Page, Yale University; Capt. H. W. Rehm, Aberdeen Proving Ground, Md.; Mr. Irwin Roman, Northwestern University; Mr.

Raleigh Schorling, Lincoln School, New York City; Mr. E. L. Thompson, Junior College, Joliet, Ill.; Dr. Bird M. Turner, University of Illinois. Four applications for membership in the Society were received.

A committee was appointed to audit the accounts of the Treasurer for the current year. A list of nominations of officers and other members of the Council was adopted and ordered printed on the official ballot for the annual election. The Treasurer of the Society to be elected at the annual meeting was made curator of all property belonging to the Society.

It was announced that the next summer meeting of the Society will be held at Wellesley College. The Mathematical Association of America, which will hold its winter meeting at Chicago, will meet with the Society at Wellesley.

The following papers were read at the October meeting:

(1) Mr. H. S. VANDIVER: "On Kummer's memoir of 1857 concerning Fermat's last theorem. Second paper."

(2) Professor R. L. BORGER: "On total differentiability."

(3) Professor ELIZABETH LESTOURGEON: "Minima of functions of lines."

(4) Professor JOSEPH LIPKA: "Complete geometric characterization of the dynamical trajectories on a surface for any positional field of force."

(5) Professor JOSEPH LIPKA: "Complete geometric characterization of brachistochrones, catenaries, and velocity curves on a surface."

(6) Professor DUNHAM JACKSON: "On the convergence of certain polynomial approximations."

(7) Dr. J. F. RITT: "On algebraic functions which can be expressed in terms of radicals."

(8) Professor A. A. BENNETT: "The Schwarz inequality for a given symmetrical convex region and given bilinear form."

(9) Professor EDWARD KASNER: "Determination of an Einstein gravitational field by means of the paths of free particles."

(10) Professor O. E. GLENN: "An algorism for differential invariant theory."

(11) Dr. T. H. GRONWALL: "Some inequalities in the theory of functions of a complex variable."

(12) Dr. W. L. G. WILLIAMS: "Fundamental systems of formal modular seminvariants of the binary cubic."

Dr. Williams' paper was communicated to the Society through Professor L. E. Dickson. In the absence of the authors the papers of Mr. Vandiver, Professor Borger, Professor LeStourgeon, Professor Lipka, Professor Jackson, Professor Glenn and Dr. Williams were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The first paper under the present title appeared in the *Proceedings of the National Academy* for May, 1920. In it, Mr. Vandiver mentions four theorems of Kummer's, which he numbers I to IV. It is shown that Kummer's arguments for each theorem are either deficient or inaccurate. In the present paper, the work of Kummer is modified in such a way as to yield proofs of theorems I and IV.

2. Professor Borger's paper contains a necessary and sufficient condition that a function of two independent variables may possess a total differential in any point where its partial derivatives exist and are finite. The extension to n variables is also indicated.

For a function of two variables this condition may be stated as follows: If the limits

$$\lim_{(k,\, h \doteq 0)} \frac{f(x+h, y+k) - f(x, y+k)}{h},$$

$$\lim_{(h,\, k \doteq 0)} \frac{f(x+h, y+k) - f(x+h, y)}{k}$$

exist; and if

$$\lim_{(k,\, h \doteq 0)} \frac{f(x+h, y+k) - f(x, y+k)}{h} = f_x,$$

$$\lim_{(h,\, k \doteq 0)} \frac{f(x+h, y+k) - f(x+h, y)}{k} = f_y,$$

the function $f(x, y)$ is totally differentiable in the point (x, y) and conversely.

The notation $\lim_{(h,\, k \doteq 0)}$ signifies that h and k approach zero so that the absolute value of the first variable in the parenthesis is less than or equal to that of the second.

3. Fréchet has defined first and second differentials for a function of a line or functional $F(\lambda)$. If $\lambda_0(x) + \eta(x)$ is a variation of $\lambda_0(x)$, the first differential is a linear functional $L(\eta)$ and the second is expressible in the form $B(\eta, \eta)$, where $B(u, v)$ is a bilinear functional in the independent arguments $u(x)$ and $v(x)$. The definitions of Fréchet apply only to functionals and differentials having continuity of order zero. Professor Le Stourgeon shows that if the continuity is of the first order, as in the case of the integrals of the calculus of variations, the differentials $L(\eta)$ and $B(\eta, \eta)$ are expressible in the forms

$$L(\eta) = \int_a^b \eta(x)du(x) + \int_a^b \eta'(x)du_1(x),$$

$$B(\eta, \eta) = \int_a^b \int_a^b \eta(x)\eta(y)d_{xy}p(x, y) + 2\int_a^b \int_a^b \eta(x)\eta'(y)d_{xy}q(x, y) + \int_a^b \int_a^b \eta'(x)\eta'(y)d_{xy}r(x, y).$$

If the functional $F(\lambda)$ has a minimum at λ_0, then it is proved that u and u_1 must satisfy an equation of the form

$$u_1(x) - \int_a^x u(x)dx = kx + l,$$

where k and l are constants. Furthermore, under certain restrictions, a necessary condition for a minimum analogous to the Jacobi condition of the calculus of variations is deduced. It is proved that when $F(\lambda_0)$ is a minimum the equation

$$\int_a^b [u(y)d_yq(y, x) + u'(y)d_yr(x, y)] - \int_a^x \int_a^b [u(y)d_yp(x, y) + u'(y)d_yq(x, y)]dx = kx + l$$

can have no solution $u(x)$, except $u(x) \equiv 0$, vanishing at $x = a$ and a point $x = x'$ between a and b.

The paper appeared in the October issue of the *Transactions*.

4. In this paper, Professor Lipka derives five geometric properties of the system of trajectories generated by the motion of a particle on any constraining surface under any

positional forces. The properties involve the system of trajectories, certain plane systems associated with them by projection into the tangent planes, the lines of force, and isothermal nets of curves, and are stated in terms of geodesic curvature, osculating and hyperosculating geodesic circles, and osculating parabolas. It is shown that the five geometric properties completely characterize the system of trajectories in the sense that any triply infinite system of curves on a surface which possesses these five properties may be considered as generated by the motion of a particle in a unique field of force.

5. This paper presents a study (analogous to that made for dynamical trajectories under positional forces) of certain systems of curves on a surface termed "n" systems—which include brachistochrones, catenaries, velocity curves, and dynamical trajectories as special cases. The field of force is conservative. Professor Lipka derives five geometric properties and shows that these completely characterize an "n" system, in the sense that any triply infinite system of curves on a surface which possesses these five properties may be considered as an "n" system—system of dynamical trajectories, brachistochrones, catenaries, velocity curves, etc., depending on the value of "n."

6. In this paper, Professor Jackson extends to a problem of polynomial approximation the method of treatment used in a paper presented to the Society at the summer meeting in Chicago, on the corresponding trigonometric case. It is assumed that $f(x)$ is a given function, continuous for $a \leqq x \leqq b$, and that $P_{mn}(x)$ is the polynomial of the nth degree which gives the best approximation to $f(x)$ over the interval, in the sense of the integral of the mth power of the absolute value of the error. The conclusions reached are not exactly parallel to those in the trigonometric case. It is found that, when m is held fast and n is allowed to become infinite, $P_{mn}(x)$ converges uniformly to the value of $f(x)$ for $a \leqq x \leqq b$, provided that $\lim_{\delta=0} \omega(\delta)/\sqrt[m]{\delta^2} = 0$, where $\omega(\delta)$ is the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leqq \delta$. The condition as stated is significant only for $m > 2$; its form is appropriately modified for $1 \leqq m \leqq 2$. Somewhat less stringent conditions are found for convergence in the interior of the interval.

7. Let $F(w, z) = 0$ be an irreducible algebraic equation of prime degree n in w. Let the number of values of z for which the Riemann surface for w has branch points be q. Dr. Ritt shows that if w can be expressed in terms of radicals, then

$$q \leqq 4 + \frac{4p}{n-1},$$

where p is the genus of the given algebraic relation.

It can be shown from this inequality that for every genus greater than zero, there is an upper bound to those prime numbers n which can serve as the degree of w in an algebraic relation which permits w to be expressed in terms of radicals.

For the special and interesting case of $p = 0$, we have $q = 2$, 3 or 4. If the monodromy group contains a cyclic substitution, we must have $q = 2$ or $q = 3$. In that case the Riemann surface for w can be changed by a linear transformation either into the surface for $z^{1/n}$ or into the surface for $\cos((\cos^{-1} z)/n)$. It follows that the only polynomials of prime degree whose inverses can be expressed in terms of radicals are $(az + b)^n + c$ and $cf_n(az + b) + d$, where $\cos nz = f_n(\cos z)$.

For $q = 4$, the equations are those which appear in the transformations of prime order of the elliptic functions.

If $q = 3$ and the monodromy group contains no cyclic substitution, we must have $n = 3r + 1$, $4r + 1$ or $6r + 1$, to each of which cases corresponds a special class of equations.

8. Starting with an arbitrary symmetrical convex region, in a domain of a linear set of elements, Professor Bennett constructs a norm, having a linearly homogeneous property, and satisfying a "triangle inequality." For a dual set of variables, an inner product and a dual norm are obtained. These will in every case satisfy the Schwarz inequality familiar in integral equations. The norms thus obtained are determined completely by the character of the given symmetrical convex region.

9. Professor Kasner shows that a four-dimensional manifold obeying Einstein's gravitational equations is essentially determined by its geodesics (paths of particles). Two Einstein manifolds cannot admit geodesic representation without being equivalent (applicable). Hence a complete knowledge of the

orbits of planets in the solar gravitational field determines that field completely, and therefore makes it possible to predict the shape of the light rays. The converse determination, by light rays alone, was discussed in one of the "Five notes on relativity" read at the summer meeting in Chicago. Thus we have in particular a two-fold connection between the motion of the perihelion of Mercury (43″), and the deflection of a light ray (1.7″).

10. The initial point of view in Professor Glenn's paper is obtained from the poles of the transformation on the differentials which one derives from relations $x_r = x_r(y_1, y_2)$ $(r = 1, 2)$ by which a differential quantic f is transformed. These are zeros of linear differential covariants df_{n-1} appertaining to a domain R defined by certain irrational expressions in the functions x_r and their derivatives. The coefficients of the expansion of f in terms of df_{n-1} as arguments are relative differential parameters, here called invariant elements, appertaining to R. Moreover every differential parameter (or covariant) is, in R, a function of the invariant elements and their derivatives, which accordingly afford a basis of classification for the known types of parameters and for several new types belonging to various domains. The methods of the paper give a direct approach to finiteness theorems and proofs in the subject. A complete system of 31 parameters of a so-called orthogonal type for the differential quantic of order six is produced in the paper, and, in addition, certain more general systems of an extended orthogonal type.

11. In this paper, Dr. Gronwall establishes inequalities analogous to that of Carathéodory for the cases where the real or imaginary part, or both, of a power series convergent in the unit circle is bounded above or below, or both ways.

12. Dr. Williams proves certain general theorems regarding formal modular seminvariants of the binary cubic, modulo p, a prime. A method of deriving a fundamental system of formal seminvariants for any particular prime is outlined and the method is applied to the cases $p = 5$ and $p = 7$. The results for $p = 5$ agree with those found by L. E. Dickson and published in his Madison Colloquium Lectures.

F. N. COLE,
Secretary.

THE MATHEMATICAL CONGRESS AT STRASBOURG.

A MATHEMATICAL congress, called as a result of the general recommendations of the Brussels conference, was held at the University of Strasbourg from September 22 to September 30, 1920. Although, as would naturally be expected, it was attended chiefly by French mathematicians, there was a fair attendance from other allied countries. There were some twelve members from the United States and six from England, but at the time of preparing this report there had appeared no printed list of those present.

In addition to the meetings of the several sections, the general programme was as follows:

Wednesday, September 22: Official opening of the congress, under the presidency of M. Alapetite, Commissaire général of the republic; address of welcome, with responses by representatives of various countries. Professor L. E. Dickson responded for this country. Visit to the university. First general session: election of president, vice-presidents, and secretary. Professor Picard was chosen president by unanimous vote. Reception to members of the congress, held in the Salle des Fêtes of the university.

Thursday, September 23: Visits to the four museums of the city. General session: address by Sir Joseph Larmor on the nature of the ether. Reception by the Society of friends of the university.

Friday, September 24: Visits to points of interest in the city. General session: address by Professor L. E. Dickson. Reception at the Hôtel-de-Ville. General meeting in honor of the congress, organized by the Société des Sciences du Bas-Rhin: address by General Tauflieb on science in Alsace; concert.

Saturday, September 25: General session: address by Professor C. J. de la Vallée Poussin. Reception at the Commissariat général.

Sunday-Monday, September 26–27: Excursions to Saint-Odile and on the Rhine.

Tuesday, September 28: General sessions: addresses by Professors Vito Volterra and N. E. Nörlund. Closing session. Banquet.

Wednesday–Thursday, September 29–30: Excursions to Saverne and Linge.

At the section meetings, held September 23–25, 27–28, the following papers were read:

SECTION I. ARITHMETIC, ALGEBRA, ANALYSIS.

YOUNG: Sur la définition de l'aire et du volume.

DICKSON: Homogeneous polynomials with a multiplication theorem.

CHÂTELET: La loi de réciprocité et les corps Abéliens.

DANIELL: On Stieltjes integrals and Volterra composition.

AMSLER: Sur le calcul symbolique sommatoire.

FUETER: Einige Sätze aus der Theorie der complexen Multiplikation der elliptischen Funktionen.

DENJOY: Sur une classe d'ensembles parfaits en relation avec les fonctions admettant une dérivée généralisée.

STOÏLOW: Sur les ensembles de mesure nulle.

DU PASQUIER: Sur une théorie des nombres complexes.

WIENER: On certain iterative properties of bilinear operations.

DRACH: L'intégration logique des équations différentielles; applications à l'analyse.

HADAMARD: Sur la solution élémentaire des équations linéaires aux dérivées partielles et sur les propriétés des géodésiques.

TAKAGI: Sur quelques théorèmes généraux de la théorie des nombres algébriques.

REY PASTOR: Sur la transformation conforme.

TYPPA: Sur les équations du troisième degré.

STÖRMER: Méthode d'intégration numérique des équations différentielles.

RÉMOUNDOS: Sur le module et les zéros des fonctions analytiques.

VAROPOULOS: Sur le module maximum des fonctions algébroïdes.

RIABOUCHINSKY: Sur le calcul des valeurs absolues.

ZERVOS: Remarques sur certaines transformations des équations aux dérivées partielles.

RADL: Sur la transformation des équations différentielles linéaires.

BOUTROUX: Sur une équation différentielle et sur une famille de fonctions entières.

LEFSCHETZ: Quelques remarques sur la multiplication complexe.

WAWRE: Sur un système d'équations à une infinité d'inconnues.

WIENER: On the theory of sets of points in terms of continuous transformations.

DE RUYTS: Une propriété simple des systèmes transformables.

OGURA: Sur la théorie de l'interpolation.

VALIRON: Sur quelques points de la théorie des fonctions entières.

ZERVOS: Sur l'intégration de certains systèmes différentiels indéterminés.

WALSH: On the location of the roots of polynomials.

ZAREMBA: Sur un théorème fondamental relatif à l'équation de Fourier.

YOUNG: Sur certaines intégrales doubles.

SAKELLARIOU: Sur les solutions discontinues du problème du calcul des variations dans l'espace à n dimensions.

SECTION II. GEOMETRY.

APRILE: Le congruenze di coniche.

BYDZOWSKY: Sur les transformations quadratiques reproduisant une quartique elliptique plane analytique.

TAYLOR: La géométrie des variables complexes.

CARTAN: Sur le problème général de la déformation.

DRACH: L'intégration logique des équations différentielles; application à la géométrie et à la mécanique.

EISENHART: Transformation des systèmes conjugués R.

SOBOTKA: Sur la deuxième indicatrice en un point d'une surface.

HOSTINSKY: Sur les propriétés de la sphère qui touche quatre plans tangents consécutifs d'une développable.

CLAPIER: Sur la transformation de Lie.

LE ROUX: Sur la géométrie des déformations des milieux continus.

EISENHART: Transformation des surfaces applicables sur une quadrique.

MURRAY: Method of classifying all polygons having a given set of vertices.

SECTION III. MECHANICS, APPLIED MATHEMATICS.

VANDERLINDEN: Les théories d'Einstein et leurs applications à l'astronomie.

BRILLOUIN: Sur un type d'action à hérédité discontinue et les équations différentielles des mouvements qui en résultent.

SCHWOERER: Détermination de l'équation séculaire de la terre dans la théorie d'Arrhenius.

GUILLAUME: Expression mono- et poly paramétrique du temps dans la théorie de la relativité.

WILIGENS: Représentation géométrique du temps dans la théorie de la relativité.

HADAMARD: Sur le problème mixte pour une équation linéaire aux dérivées partielles.

BANERJI: Some problems in earthquakes.

BOCCARDI: Sur le déplacement du pôle.

DA COSTA-LOBO: Sur la courbe décrite par le pôle sur la surface de la terre.

BOCCARDI: Sur les approximations numériques et les sciences d'observation.

FARID-BOULAD: Nouveau théorème pour calculer les tensions des barres surabondantes des poutres et arcs à montants et croix de Saint André.

GREENHILL: La fonction potentielle uniaxiale et sa fonction de force orthogonale.

HATZIDAKIS: Sur quelques formules de géométrie cinématique.

HOSTINSKY: Sur un problème général de la mécanique vibratoire.

MAILLARD: Mise au point des hypothèses cosmogéniques-nébulaires.

ROSENBLATT: Sur la théorie des figures d'équilibre des masses fluides animées d'un mouvement de rotation.

RIABOUCHINSKY: Sur la résistance des fluides.

LARMOR: Sur les pressions des ondes sonores.

GULDENBERG: Une application des polynômes d'Hermite à un problème statistique.

DE DONDER: Sur la gravifique.

LARMOR: Sur les rayons diffractés attachés aux images optiques.

BARRAU: Sur la cinématique plane.

BAUER: Remarques élémentaires sur le principe de relativité en electrodynamique.

SECTION IV. PHILOSOPHY, HISTORY, PEDAGOGY.

GÉRARDIN: Décomposition des nombres. Machines à congruences. Des nombres entiers. Jeux scientifiques.

BROCARD: 24 propositions de Fermat.

DELAPORTE: Sur la réforme du calendrier.

DU PASQUIER: Sur les nombres transfinis.

GREENHILL: Les fonctions de Fourier et Bessel comparées.

D'OCAGNE: La pratique courante de la méthode nomographique des points alignés: à propos de ses applications de guerre.

GROSSMANN: Sur l'état de publication des œuvres d'Euler.

POSTIGLIONE: Cyclómetrie mécanique.

DUBECQ: Communication sur l'enseignement, République Argentine.

ZERVOS: Sur l'enseignement mathématique.

DAVID EUGENE SMITH.

NOTE ON A METHOD OF PROOF IN THE THEORY OF FOURIER'S SERIES.

BY PROFESSOR DUNHAM JACKSON.

(Read before the American Mathematical Society September 7, 1920.)

IT has been pointed out on various occasions* that if $f(x)$ is a continuous function of period 2π satisfying the Lipschitz-Dini condition, that is, if $\lim_{\delta=0} \omega(\delta) \log \delta = 0$, where $\omega(\delta)$ is the maximum of the oscillation of $f(x)$ in an interval of length δ, then the uniform convergence of the Fourier series for $f(x)$ can be inferred almost immediately from the following two propositions:

A.† If $f(x)$ satisfies the Lipschitz-Dini condition,‡ there exists for every positive integral value of n a finite trigonometric sum $\tau_n(x)$, of order n at most, such that $\lim_{n=\infty} r_n \log n = 0$, where r_n is the maximum of $|f(x) - \tau_n(x)|$.

* Cf., e.g., Lebesgue, "Sur les intégrales singulières," *Annales de la Faculté de Toulouse*, series 3, vol. 1 (1909), pp. 25–117; pp. 116–117.

† Cf., e.g., Lebesgue, loc. cit., p. 116; D. Jackson, "On the approximate representation of an indefinite integral, etc.," *Transactions Amer. Math. Society*, vol. 14 (1913), pp. 343–364; p. 350.

‡ It is understood throughout the paper that every function considered has the period 2π.

*B.** If $\varphi(x)$ is any continuous† function, and $S_n(x)$ the partial sum of the Fourier series for $\varphi(x)$ to terms of order n, then $|S_n(x)|$ can not exceed $KM \log n$, where M is the maximum of $|\varphi(x)|$, and K is an absolute constant.

The central point in the proof is the fact that $\tau_n(x)$ is identical with the partial sum of its own Fourier series to terms of order n. It is the purpose of this note to show that similar reasoning can be applied to the arithmetical mean of Fejér,‡ in spite of the fact that the Fejér mean formed for a finite trigonometric sum $\tau_n(x)$ is not the same as $\tau_n(x)$. It is necessary to change the argument somewhat, but there is no difficulty in making the required modification.

Let $f(x)$ be now an arbitrary continuous function, and let $\sigma_n(x)$ be the arithmetical mean of the partial sums of the Fourier series for $f(x)$, to terms of order n. The uniform convergence of $\sigma_n(x)$ to the value $f(x)$ is to be deduced from the propositions:

C. (Weierstrass's theorem.)§ If $f(x)$ is continuous, there exists for every positive integral value of n a finite trigonometric sum $\tau_n(x)$, of order n at most, such that $\lim_{n=\infty} r_n = 0$, where r_n is the maximum of $|f(x) - \tau_n(x)|$.

D.‖ If $\varphi(x)$ is any continuous function (more generally, any integrable function), and $\sigma_n(x)$ the Fejér mean of the Fourier series for $\varphi(x)$ to terms of order n, then $|\sigma_n(x)|$ can not exceed M, where M is the maximum of $|\varphi(x)|$.

Let ϵ be any positive quantity. Let a finite trigonometric sum $\tau_p(x)$, of order p, be determined, according to Proposition *C*, so that

$$|f(x) - \tau_p(x)| < \tfrac{1}{3}\epsilon.$$

Let $\sigma_{n1}(x)$ be the Fejér mean, of order n, for the function $\tau_p(x)$, and $\sigma_{n2}(x)$ the corresponding mean for the function $f(x) - \tau_p(x)$.

* Cf., e.g., Lebesgue, loc. cit., p. 116; D. Jackson, "On approximation by trigonometric sums and polynomials," *Transactions Amer. Math. Society*, vol. 13 (1912), pp. 491–515; pp. 502, 512–515.

† It is sufficient for the truth of the statement that φ be integrable, but for present purposes there is no need of speaking of any but continuous functions.

‡ Fejér, "Untersuchungen über Fouriersche Reihen," *Mathematische Annalen*, vol. 58 (1904), pp. 51–69.

§ Weierstrass, "Ueber die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen," *Berliner Sitzungsberichte*, 1885, pp. 633–639, 789–805; p. 801.

‖ Fejér, loc. cit., p. 60.

Then

$$\sigma_n(x) = \sigma_{n1}(x) + \sigma_{n2}(x). \tag{1}$$

By Proposition *D*,

$$|\sigma_{n2}(x)| < \tfrac{1}{3}\epsilon,$$

$$|f(x) - \tau_p(x) - \sigma_{n2}(x)| < \tfrac{2}{3}\epsilon, \tag{2}$$

for all values of n. The quantity $\sigma_{n1}(x)$ is the arithmetical mean of $n+1$ finite trigonometric sums, of which all from the $(p+1)$th on, if $n \geqq p$, are identical with $\tau_p(x)$, while each of the first p is composed of a part of the terms of $\tau_p(x)$. Added together, the first p sums which enter into the mean give a finite trigonometric sum $\omega_{p-1}(x)$, which is of order $p-1$ at most, and independent of n. So $\sigma_{n1}(x)$ can be written in the form

$$\sigma_{n1}(x) = \frac{\omega_{p-1}(x) + (n+1-p)\tau_p(x)}{n+1}$$

$$= \tau_p(x) + \frac{\omega_{p-1}(x) - p\tau_p(x)}{n+1}.$$

As the last numerator is independent of n, $\sigma_{n1}(x)$ approaches $\tau_p(x)$ uniformly as n becomes infinite—a fact which is fairly obvious in the first place—and, if n is sufficiently large,

$$|\tau_p(x) - \sigma_{n1}(x)| < \tfrac{1}{3}\epsilon. \tag{3}$$

By combination of (1), (2), and (3), for values of n satisfying (3),

$$|f(x) - \sigma_n(x)| < \epsilon,$$

which completes the convergence proof.

THE UNIVERSITY OF MINNESOTA,
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ON IMPLICIT FUNCTIONS.

BY MR. F. H. MURRAY.

METHODS of solving a system of m equations in $n > m$ variables, by reducing the problem to that of solving a set of differential equations, have already been given;* the aim of this paper is to present a method by which the system of m equations can be reduced immediately to a system of m differential equations of the first order, from which reduction the existence theorem and a method of constructing the solutions follow directly. For simplicity the case of two equations in four variables will be treated.

§ 1. *Reduction to a System of Differential Equations.*

It is required to solve the system

$$f_1(x, y, u, v) = 0, \quad f_2(x, y, u, v) = 0 \tag{1}$$

in the neighborhood of a set of values (x_0, y_0, u_0, v_0) for which

$$f_1(x_0, y_0, u_0, v_0) = 0, \quad f_2(x_0, y_0, u_0, v_0) = 0.$$

There is no loss in generality in assuming $x_0 = y_0 = u_0 = v_0 = 0$; suppose x, y to be the independent, u, v the dependent variables. It will be assumed that all the first partial derivatives exist and are continuous in the neighborhood of the origin defined by the inequalities

$$\sqrt{x^2 + y^2} \leqq a, \quad |u| \leqq b, \quad |v| \leqq b. \tag{R}$$

Also, assume that the Jacobian

$$\Delta = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}$$

does not vanish at the origin (0, 0, 0, 0); consequently the constants a, b can be so chosen that $\Delta \neq 0$ in R. Introduce

* Horn, Gewöhnliche Differentialgleichungen beliebiger Ordnung.

polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Equations (1) become

(1′) $f_1(r\cos\theta, r\sin\theta, u, v) = 0, \quad f_2(r\cos\theta, r\sin\theta, u, v) = 0.$

If functions $u(r, \theta)$, $v(r, \theta)$, satisfying (1′) can be found, such that $\partial u/\partial r$, $\partial v/\partial r$ are defined throughout the region R, we must have

$$(2)\qquad \begin{aligned} &\frac{\partial f_1}{\partial x}\cos\theta + \frac{\partial f_1}{\partial y}\sin\theta + \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial f_1}{\partial v}\frac{\partial v}{\partial r} = 0, \\ &\frac{\partial f_2}{\partial x}\cos\theta + \frac{\partial f_2}{\partial y}\sin\theta + \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial f_2}{\partial v}\frac{\partial v}{\partial r} = 0. \end{aligned}$$

Since $\Delta \neq 0$,

$$(3)\qquad \begin{aligned} \frac{\partial u}{\partial r} &= \frac{\begin{vmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial x} \end{vmatrix}\cos\theta + \begin{vmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial y} \end{vmatrix}\sin\theta}{\Delta}, \quad u(0) = 0, \\ \frac{\partial v}{\partial r} &= \frac{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial u} \end{vmatrix}\cos\theta + \begin{vmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial u} \end{vmatrix}\sin\theta}{\Delta}, \quad v(0) = 0. \end{aligned}$$

In these equations consider θ as a parameter, u, v, r as the variables; equations (3) can be written in the form

$$(3')\qquad \frac{du}{dr} = A(u, v, r), \quad \frac{dv}{dr} = B(u, v, r), \quad u(0) = v(0) = 0.$$

Suppose the original functions f_1, f_2 such that for every θ, $0 < \theta \leqq 2\pi$, there exist constants C, D if u', u'', v', v'', are in R, such that

$$(4)\qquad \begin{aligned} &|A(u'', v'', r) - A(u', v', r)| < C|u'' - u'| + D|v'' - v'|, \\ &|B(u'', v'', r) - B(u', v', r)| < C|u'' - u'| + D|v'' - v'|. \end{aligned}$$

§ 2. *Final Solution.*

Suppose that for $u = v = 0$, $r \leqq h$, $|A| \leqq M_0$, $|B| \leqq M_0$; for u, v, r in R, $|A| \leqq M$, $|B| \leqq M$. Then from the existence theorem for ordinary differential equations,* if $C + D = K$, the equations (3′) have solutions u, v if $0 \leqq r \leqq \delta$, where δ is the larger of the quantities δ_1, δ_2,

$$\delta_1 \leqq a \leqq \frac{b}{M}, \quad \delta_2 \leqq a \leqq \frac{1}{K}\log\left(1 + \frac{bK}{M_0}\right).$$

The functions u, v satisfy (3′), hence have derivatives with respect to r.

Since A, B are periodic in θ, the constants C, D have upper limits independent of θ, consequently K, M_0 can be defined independent of θ; the series defining u, v converge uniformly in θ, hence u, v are continuous functions of θ. The constant M can also be defined independent of θ, with the result that the solutions u, v are continuous functions of r, θ, and differentiable with respect to r, if $0 \leqq r \leqq \delta$.

Introduce these functions in (1′); since

$$\frac{\partial f_1}{\partial r} = \frac{\partial f_2}{\partial r} = 0, \quad f_1 = \phi_1(\theta), \quad f_2 = \phi_2(\theta).$$

But within the region $0 \leqq r \leqq \delta$, f_1, f_2 are continuous functions of r, θ; for $r = 0, x = y = u = v = 0$, consequently

$$\phi_1(\theta) = 0, \quad \phi_2(\theta) = 0.$$

These equations must hold for every θ, hence are identities, and equations (1′) are identically satisfied. The functions u, v are the solutions required.

For n variables $x_1, x_2, \cdots, x_n$,

$$r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}, \qquad \cos\theta_i = x_i/r.$$

* Picard, Traité d'Analyse, 2, pp. 343, 345.

ON A PENCIL OF NODAL CUBICS. SECOND PAPER.

BY PROFESSOR NATHAN ALTSHILLER-COURT.

(Read before the American Mathematical Society April 24, 1920.)

1. REFERRING to the notations of my first communication,* let Γ_n, Γ_s be two cubics of the pencil Γ, and let X_n, X_s, F be the points of intersection of a fixed line l through the double point O with Γ_n, Γ_s, and the basic line ABC respectively; let X_n', X_s', F' be the analogous points on any other line l' through O. The lines X_nX_n', X_sX_s', meet on ABC according to proposition 8 of the paper referred to, and therefore

$$(OFX_nX_s) = (OF'X_n'X_s').$$

Thus a variable line l' through the double point O meets the two cubics Γ_n, Γ_s in two points X_n', X_s' which, with the double point O and the trace F' of l' on ABC, form an anharmonic ratio having a constant value. Consequently:†

Two nodal cubics having in common three collinear points, the double point, and the tangents at this point, are homological.

The double point and the base of the three collinear points are respectively the center and the axis of homology.

Two tricuspidal quartics having in common three concurrent tangents, the double tangent, and the two points of contact with this line, are homological.

The double tangent and the common point of the three concurrent tangents are respectively the axis and the center of homology.

2. Any two cubics of the pencil Γ are thus homological, i.e., Γ is a pencil of homological cubics. This fundamental property of Γ furnishes immediately a second proof of most of the propositions of my first paper. It also brings to light many new propositions. For instance:

* This BULLETIN, vol. 26, p. 203 (February, 1920).
† M. Chasles, Traité de Géométrie supérieure, 2d edition, p. 350.

(*a*) The triangle formed by the three inflexional tangents of any cubic of the pencil Γ is homological to the corresponding triangle of any other cubic of the pencil, the double point O being the center of homology, hence: *The vertices of the triangle formed by the inflexional tangents of a variable cubic of the pencil Γ describe three straight lines passing through the double point.*

(*b*) It is known that the lines joining the pairs of corresponding points on a nodal cubic Γ_n envelop a conic ω_n, which is the Cayleyan of the cubic of which Γ_n is the Hessian. The conics (ω) corresponding to ω_n in the various curves of the pencil Γ are homological to ω_n. Hence: *The conics (ω) of the cubics of the pencil Γ have two points in common on the basic line of the pencil.*

It should be noticed however that the conics (ω) do not, in general, form a pencil. Being homological, in order to form a pencil, they would have to be tangent to the lines projecting from the center of homology O the two points that the conics have in common with the axis of homology ABC* at these points. Now it is known that the tangents from O to ω_n are the tangents OT_1, OT_2 to Γ_n at O, their points of contact with ω_n being the points determined on OT_1, OT_2 by the inflectional line i_n of Γ_n.

3. In the cases when the common elements of the two curves of section 1 are of some special nature or are taken in some special position the mutual relation of the two curves may become of particular interest. For lack of space we shall give only one example.

The tricuspidal hypocycloid (or tricusp) is a quartic touching the line at infinity at the cyclical points, according to a proposition due to Cremona.† If we consider two such curves having three concurrent tangents in common, their axis of homology will be at infinity. Consequently: *Two tricuspidal hypocycloids having three concurrent tangents in common are similar and similarly placed. The point of intersection of their common tangents is the center of similitude of the two curves.*

UNIVERSITY OF OKLAHOMA,
February, 1920.

* C. Servais, "Sur les faisceaux de coniques," sec. 9, *Le Matematiche pure ed applicate*, vol. I (December, 1901).

† See R. C. Archibald, "The cardioid and the quartics with three cusps," *Annals of Mathematics*, 2d series, vol. 4 (1903).

HAUSDORFF'S GRUNDZÜGE DER MENGENLEHRE.

Grundzüge der Mengenlehre. By FELIX HAUSDORFF. Veit and Company, Leipzig, 1914. viii + 476 pp.

IF there are still mathematicians who hold the theory of aggregates under general suspicion, and are reluctant to grant it full recognition as a rigorous, mathematical discipline, they will find it hard to retain their doubts under fire of the logic of Hausdorff's treatise. It would be difficult to name a volume in any field of mathematics, even in the unclouded domain of number theory, that surpasses the Grundzüge in clearness and precision.

But it is only in a subsidiary rôle that the Grundzüge is an answer to the skeptics. Its most striking feature is that it is a work of art of a master. No one thoroughly acquainted with its contents could fail to withhold admiration for the happy choice and arrangement of subject matter, the careful diction, the smooth, vigorous and concise literary style, and the adaptable notation; above all things, however, for the highly pleasing unifications and generalizations and the harmonious weaving of numerous original results into the texture of the whole.

It is not an uncommon fault of authors of treatises on general subjects to expound their own researches with an unwarranted degree of detail; so that at times, if one has no other evidence, one may be rightly led to suspect particular portions, on account of their remoteness from the central ideas, of being the author's own handiwork. This fault is not shared by the Grundzüge. Few treatises on as comprehensive a subject as the theory of aggregates contain as large a proportion of the author's investigations; yet the parts that are distinctly Hausdorff's own contributions are properly inserted in view of their generality and in relation to other topics.

The author is endowed with a keen psychological and didactic instinct that prompts him to depart from his usual succinctness when engaged in the clarification of the more important ideas. The following quotation may serve to illustrate this pedagogic sense and the general lucidity of style, not without an occasional glimmer of humor. After speaking

of the primitive stage of numerical comparison—by means of direct, successive mating—of a pile of apples and a pile of pears, he says (page 45): "But if the apples and the pears are in different places, and the transportation of one pile to the other is attended with difficulties, the inventive mind of man will in the next stage make use of an intermediary set of conveniently transportable objects, such as stones, shells, or chips, and infer the equivalence $A \sim B$ from the equivalences $A \sim C$, $B \sim C$. Finally, however, even this earthly residuum will be eliminated, and the intermediary set will be replaced by a system of spoken, written, or thought symbols, —the number symbols, 1, 2, $\cdots$. *Comparison* turns into *counting*, and equivalent sets now acquire a common property, the number of their elements.

"These remarks, for which no claim whatsoever is entered on psychological or cultural-historical grounds, are intended merely to make clear that equivalence is the natural foundation for the comparison of aggregates, and that by its means we may undertake even the seeming paradox of counting infinite sets."

That the author enjoyed himself while at work may be seen from such passages as the following (page 61): "From an 'alphabet,' i.e. a finite set of 'letters,' we may construct a countable assemblage of finite complexes [= ordered sets] of letters, i.e. 'words,' among which, of course, meaningless words such as abracadabra occur. If in addition to the letters, other elements are used, such as punctuation marks, type-spacings, numerals, notes, etc., we see that the assemblage of all books, catalogs, symphonies and operas is also countable, and would remain countable even if we were to employ a countable set of symbols (but for each complex only a finite number). On the other hand, if in the case of a finite number of symbols we restrict the complexes to a maximum number of elements, agreeing for example, to rule out words of more than one hundred letters and books of more than one million words, these assemblages become finite; and if we assume with Giordano Bruno an infinite number of heavenly bodies, with speaking, writing, and musical inhabitants, it follows as a mathematical certainty that on an infinite number of these heavenly bodies there will be produced the same opera with the same libretto, the same names of the composer, the author of the text, the members of the orchestra and the singers."

Several other examples of spirited and colorful language are the following. Speaking of the equivalence of the whole and the part of an infinite set, he says (page 34): "Of course, when asserted in the rather provocative form, A has as many elements as B, it is one of those 'paradoxes of the infinite' that shock the unprepared mind." Again, page 34: "A segment and an arbitrarily small partial segment, a kilometer and a millimeter, the sun's globe and a drop of water have in this sense the 'same number of points.'" Page 48: ". . . we shall have to desist from giving every proper subset a cardinal number $< \mathfrak{a}$; we must violate the hallowed axiom 'totum parte majus,' as we must in general expect that calculation with infinite cardinals will deviate in many respects from that with finite cardinals, without thereby espying the minutest objection against these infinite numbers." On page 63, after learning that $10^{\aleph_0} = \aleph$, we are led to the equations $\aleph = 2^{\aleph_0} = 3^{\aleph_0} = \cdots$ by the remark that "the fact that we have ten fingers is obviously without influence on the theory of aggregates." Page 60: "The equivalence of the set of whole numbers with the much more inclusive set of rational numbers belongs to those facts of the theory of aggregates which impress you on your first acquaintance with them as astonishing and even paradoxical; especially if you have before your eyes the geometric representation (of the correspondence between the numbers and the points of a straight line), and picture to yourself on the one hand, the 'integral' points, which lie isolated at finite distances from one another, and on the other hand, the 'rational' points, which are distributed over the entire line as dust of more than microscopic fineness."

One of the characteristic traits of the style is its continuity, brought about by the neat and confident conjunctional devices of the author. On page 335 there is need of the awful descent from the general spaces previously considered to the very special euclidean plane. The author is unwilling to take the plunge without assuring himself of the reader's good-will: "In the subsequent discussion we prefer to confine ourselves to the plane. The extension to three or more dimensional space presents in part not inconsiderable difficulties, because the rôle which polygons play in the plane falls there to the lot of the much less simple polyedra and hyper-polyedra. Even in the plane we shall find that the apparently most

plausible, intuitive assertions require fairly complicated proofs. A certain prolixity is already produced by the fact that sets that are not compact, and are therefore unbounded, behave in many respects not like bounded sets, and indeed, less regularly. A radical remedy would be the adjunction of a point 'at infinity,' as in function theory; but thereby the character of a metrical space is destroyed, and if again, you get rid of this evil by means of stereographic projection on the sphere, you lose in exchange certain elementary geometric advantages of the plane. We must, therefore, come to terms as best we can with the 'dreary infinities of homaloidal space' as Clifford says."

We now turn to a more detailed description of the contents. In view of the great wealth of ideas—made possible by the concentrated style—we must refrain from attempting to discuss even all the important topics; only some of these can we describe in detail, and we shall give preference to the more novel or the less technical features.

The book is fittingly inscribed to Georg Cantor, "the creator of the theory of aggregates."

Chapter I (pages 1–31) deals with the sum (denoted by $\mathfrak{S}$), section (Durchschnitt, denoted by $\mathfrak{D}$), and difference of sets. Of the numerous topics treated in this chapter, we shall discuss the notion of aggregate, the principle of duality, and symmetric sets. Among the other topics may be mentioned difference chains, rings and fields (Ringe und Körper), sequences of sets and of real numbers, σ- and δ-systems, and the non-convergence points of a sequence of functions.

Aggregate is defined in the cantorean naïve fashion—as distinguished from the less debatable, but more restraining manner of Zermelo—as a whole constituted by the conceived assembling of individuals. The author makes it clear that it is inadvisable on pedagogical and other grounds to found everything upon Zermelo's Grundlagen; paradoxes are duly banished, however, by appropriate interpretation of the naïve definition (see page 106 for the disposal of the Burali-Forti antinomy).

If $A_1, A_2, \cdots$ are subsets of a set M and $\overline{A}_1 = M - A_1$, $\overline{A}_2 = M - A_2, \cdots$ their complements,—we denote generally by $\overline{X}$ the complement of X in M—, then the complement in M of the sum of the given sets is the section of their com-

plements $[\overline{\mathfrak{S}}(A_1, A_2, \cdots) = \mathfrak{D}(\overline{A}_1, \overline{A}_2, \cdots)]$ and the complement of the section of the given sets is the sum of their complements $[\overline{\mathfrak{D}}(A_1, A_2, \cdots) = \mathfrak{S}(\overline{A}_1, \overline{A}_2, \cdots)]$. Since $P = Q$, $P \subset Q$ (i.e., P is a subset of Q) imply $\overline{P} = \overline{Q}$, $\overline{P} \supset \overline{Q}$, it follows that every equation between sets remains true if every set is replaced by its complement and the symbols $\mathfrak{S}$ and $\mathfrak{D}$ are interchanged; and every inequality remains true after the same changes and the additional interchange of $\supset$ and $\subset$. For example, $A \subseteqq \mathfrak{S}(A, B)$ leads to $\overline{A} \supseteqq \mathfrak{D}(\overline{A}, \overline{B})$ and hence to $A \supseteqq \mathfrak{D}(A, B)$. This simple property Hausdorff calls the "principle of duality"; he utilizes it in various connections to secure results through formulas usually obtained otherwise. We remark here that the author uses formulas to a much larger extent than is customary in the theory of aggregates, one of whose noticeable characteristics is the unusual freedom from calculational methods.

Let $X_1, X_2, \cdots, X_m$ be m given sets. Taking a cue from algebra, we seek a list of simple sets which like the sum $\mathfrak{S}(X_1, X_2, \cdots, X_m)$ and the section $\mathfrak{D}(X_1, X_2, \cdots, X_m)$ involve the given sets symmetrically. Hausdorff calls these sets "symmetrische Grundmengen," and defines them as follows: $A_i (i = 1, 2, \cdots, m)$, the ith such set, consists of the elements that occur in at least i X's. Thus $A_1 = \mathfrak{S}(X_1, X_2, \cdots, X_m)$, $A_m = \mathfrak{D}(X_1, X_2, \cdots, X_m)$. An essential property of these sets is that they are expressible in terms of the X's by means of sums and sections. Their introduction is due to Hausdorff. The author considers various properties of these sets, and in particular, utilizes them in an interesting theory of measure, which was at first planned as final but was later discarded for a more concise treatment; a sketch of the old theory appears in the appendix.

The second chapter (pages 32–45) deals with functions, products, and powers, and their laws of operation.

The third chapter (pages 45–69) treats of the cardinal numbers. Cardinal number is not defined with Cantor as what remains of a set after the individual nature and the order of the elements are abstracted; nor with Russell, as a class of classes. Hausdorff takes the simple and formal point of view, which is clearly the most satisfactory: We associate uniquely with a system of sets A a system of things a—of indifferent nature—in such a way that the same things corre-

spond to two sets if and only if the sets are equivalent. These things or symbols we call cardinal numbers. Two proofs are given of the Bernstein equivalence theorem, the first essentially like that of Bernstein, and the second, according to Zermelo, without the use of the infinite set of integers. The rest of the chapter is devoted to the comparison of cardinal numbers, and includes such theorems as $\aleph^{\aleph_0} = \aleph$, $\aleph_0^{\aleph} = 2^{\aleph} > \aleph$ ($\aleph$ = cardinal number of the continuum), and the theorem of J. König. Famous theorems, such as those of the non-denumerability of the continuum, and of the equivalence of the plane of points and the line of points, frequently appear—as is also the case in other chapters—as very special cases of general considerations, or as side remarks in the current text. For the non-denumerability of the continuum, however, a special proof is added.

Chapter IV (pages 69–101) takes up order; a substantial portion of the ideas and results is due to Hausdorff. After various definitions of simple (= linear) order, the sum of an ordered set (the "argument") of any number of ordered sets, and the product of a finite number of sets are defined, and the laws for operating with these processes given. The subset M of the ordered set A is said to be "coinitial" with A if no element of A exists preceding every element of M. Similarly, "cofinal." M is "dense in A" if for every pair $a < b$ of elements of A there exists a pair of elements $m < n$ of M such that $a \leqq m < n \leqq b$. The decomposition $A = P + Q$, where $P(\neq 0)$ and $Q(\neq 0)$ have no elements in common and every element of P precedes every element of Q, is said to be a "jump" (Sprung) if P has a last element and Q a first; a "gap" (Lücke), if neither P has a last, nor Q a first. A "dense set" (in the "absolute," as contrasted with the "relative" sense) is one without jumps; a "continuous set," one without jumps or gaps. A "scattered set" is one possessing no dense subset. A sum $\Sigma_i^J A_i$ $(A_i \neq 0)$ is scattered when and only when the argument J is scattered and each A_i is scattered. Every ordered set is either scattered or the sum of scattered sets over a dense argument. The chapter closes with the discussion of types of order, in particular, of the class of countable dense types, and of continuous types.

The fifth chapter (pages 101–139) is devoted to normally ordered (wohlgeordnete) sets and the ordinal numbers. The treatment comprises comparability of cardinal numbers, trans-

finite induction, powers and products, alephs and the number classes, the initial numbers (Anfangszahlen) and Zermelo's Wohlordnungssatz. The proof of the general theorem $\aleph_a\aleph_a = \aleph_a$ for every ordinal number a, is given in elegant and brief form; the first proof of the theorem, given by Hessenberg in his Grundbegriffe der Mengenlehre, is long and roundabout. It may be remarked that frequently the simplest and most elegant proofs of important theorems are to be found in the Grundzüge, either directly or after appropriate modification of the generalized form in which they usually appear.

For the Wohlordnungssatz both of Zermelo's proofs with unessential but neat modifications are given. Hausdorff has no difficulty—neither has the reviewer—in accepting either of these proofs as rigorous. In fact, as is sometimes the case with the work of mathematicians who have misgivings about the theorem, the multiplicative principle (Prinzip der Auswahl) steals in noiselessly (cf. for example, page 54) before the Wohlordnungssatz is mentioned.

The sixth chapter (pages 139–209) contains a wealth of material mostly from the author's own researches. Unfortunately space will not permit—especially because of the more technical character of the subject matter—a description of these elegant and general results. We must content ourselves merely with mentioning the partially ordered sets, the distinction as related to coinitiality and cofinality of the element and gap characteristics and the consequent classification of ordered sets, the general products and powers of ordered sets and the interesting connection with non-archimedean number systems, as shown by the general theorem of Hahn (*Berichte der Wiener Akademie der Wissenschaften*, 1907).

The remaining chapters of the book (VII–X) will prove of more general interest because they are concerned with the applications of the abstract theory to the study of space relations. It is in these chapters especially that Hausdorff impresses you with his masterful exposition. The theory of point sets is cast into a new and more general mold, and the resulting treatment is characterized throughout by originality, naturalness, and beauty.

Chapter VII (pages 209–260) begins with the statement: "The theory of aggregates has celebrated its most beautiful triumphs in its application to point sets and in the clarification and heightened precision of the fundamental concepts of

geometry; this is admitted even by those who demean themselves skeptically towards the abstract theory of aggregates." The subject matter of the chapter concerns point sets in general spaces. The author's justification of his abstract treatment is as follows (page 210): "Now a theory of spatial point sets would naturally have, in virtue of the numerous accompanying properties, a very special character, and if we wished to confine ourselves from the outset to this single case, we should be obliged to develop one theory for linear point sets, another for planar point sets, still another for spherical point sets, etc. Experience has shown that we may avoid this pleonasm and set up a more general theory comprehending not only the cases just mentioned but also other sets (in particular, Riemann surfaces, spaces of a finite or an infinite number of dimensions, sets of curves, and sets of functions). And, indeed, this gain in generality is associated not with increased complication, but on the contrary, with a considerable simplification, in that we utilize—at least for the leading features—only few and simple assumptions (axioms). Finally we secure ourselves in this logical-deductive way against the errors into which our so-called intuition may lead us; this alleged source of knowledge—the heuristical value of which, of course, no one will impugn—has, as it happens, shown itself so frequently insufficient and unreliable in the more subtle parts of the theory of aggregates, that only after careful examination may we have faith in its apparent testimony."

Hausdorff does not bind himself to a single set of assumptions. The center of interest lies, of course, in the theorems, and the assumptions are graded accordingly, a new assumption or a modification being adopted only when the mathematical content naturally calls for such a change. In the carefully planned march from the abstract in the direction of greater specialization, Hausdorff gives repeated evidence of his mathematical-esthetic insight.

The developments in Chapter VII are based entirely upon the following "neighborhood" postulates (Umgebungsaxiome, page 213). A "neighborhood" is a point set. The abstract set or space E in question is unrestricted except for the postulates:

(A) To every point x of E there corresponds at least one neighborhood U_x; every neighborhood U_x contains the point x.

(B) If U_x, V_x are two neighborhoods of x, there is a neighborhood W_x contained in both.

(*C*) If the point y lies in U_x, there is a neighborhood U_y lying in U_x.

(*D*) For $x \neq y$ there are two neighborhoods U_x, U_y with no point in common.

For the euclidean plane, the neighborhoods of a point P may be taken as the circles (exclusive of the boundary) having P as center.

A space satisfying the four neighborhood postulates is called "topological."

There are numerous concrete examples of topological spaces, among which may be mentioned the ordinary euclidean spaces (also after adjunction of the ideal point at infinity), certain spaces in which the distance from point to point is measured on a non-archimedean scale, space of a denumerable infinity of dimensions, and function space. Some of these spaces are also "metric" (see below).

An "inner point" of a set A—belonging to the entire space or "universe" E—is one possessing a neighborhood lying entirely in A. A point of A that is not one of its inner points is a "brink" point (Randpunkt), as distinguished from "boundary" point (Grenzpunkt), which need not belong to the given set. A "region" (Gebiet) is a set every point of which is an inner point of the set; a "brink aggregate" (Randmenge), one every point of which is a brink point. The inner points of the complement B of A ($A + B = E$) are called the "outer" points of A; the boundary points of A consist of the brink points of both A and B. The universe E and every neighborhood is a region. The inner points of an arbitrary set constitute a region; the brink points, a brink aggregate. The sum of any number, and the section of a finite number, of regions are regions.

Connected with the last statement, there is a simple but fruitful principle: If a system of sets M—like the system of regions—has the property that the sum of any number of sets of the system belongs to the system, we may, for any given set A containing at least one M as subset, define the largest M contained in A; if the section of any number of M's is an M, we may for every A contained in at least one M define the smallest M containing A. This principle is used in various connections, for example, in the definition of the "kernel" (see below).

The introduction of the sets A_α, A_β, A_γ—the first has been

little used—leads to considerable formal simplification, and enables Hausdorff to give many short proofs through formal processes instead of direct reflection upon the nature of the hypotheses. x is an α-point of A if every neighborhood U_x contains at least one point of A (which may be x itself); a β-point, if every neighborhood U_x contains an infinite number of points of A; a γ-point, if every neighborhood U_x contains a non-denumerable set of points of A. A_α, A_β, A_γ are the respective totalities of these points (A_β, the derivative of A). That A is "closed" may be expressed by $A \supseteq A_\beta$, or by $A = A_\alpha$; "dense-in-itself," by $A \subseteq A_\beta$ or by $A_\alpha = A_\beta$; "perfect," by $A = A_\beta$. The sets A_α, A_β, A_γ are closed. The section of any number of closed sets and the sum of a finite number of closed sets are closed. The sum of any number of sets each dense-in-itself is dense-in-itself. The largest subset of A that is dense-in-itself exists according to the principle just mentioned; it is called the "kernel" of A.

An infinite set without β-points is said to be "divergent"; a set without divergent subsets, "compact." The set A converges to the limit x if every neighborhood of x contains all the points of A with the possible exception of a finite number. A decreasing sequence $A_1 \supseteq A_2 \supseteq \cdots$ of compact, closed, non-vanishing sets has a non-vanishing section (Cantor). A compact, closed set contained in the sum of a sequence of regions is contained in the sum of a finite number of these regions (Borel).

A clear and systematic treatment is given of the limits of a sequence of sets $\{A_n\}$. Six different kinds of limits are distinguished: (1) the "lower limit" consists of the points belonging to "nearly all" the A_n, i.e., all with the possible exception of a finite number; (2) the "upper limit," of the points belonging to an infinity of the A_n; (3) the "lower closed limit," of the points (belonging to the A_n or not) every neighborhood of which contains points of nearly every A_n; (4) the "upper closed limit," of the points every neighborhood of which contains points of an infinity of the A_n; (5) the "lower limit region," of the points for which a neighborhood exists belonging (in its entirety) to nearly all the A_n; and (6) "the upper limit region," of the points for which a neighborhood exists belonging to an infinite number of the A_n. Various modes of representation of these limits by means of sums and sections are given.

Another novel feature is the systematic introduction and use of the notion of "relativity." A is said to be (relatively) "closed in M" if it is the section of M and a closed set; a "relative region of M," if it is the section of M and a region. These notions are special cases of a complete theory of relativity, which arises by substituting for the universe E an arbitrary subset M of it. The neighborhoods U_x are replaced by their sections with M, which again satisfy the neighborhood assumptions; M may therefore be regarded as a new universe possessing all the properties of topological spaces. We thus have relative α-points, relative inner points, and so on.

The definition of connectivity differs from those heretofore given, but it is the most desirable in the opinion of the reviewer: A non-vanishing set M is said to be "connected" if it is not expressible as the sum of two sets ($\neq 0$) that are (relatively) closed in M and have no points in common. A "component" of a non-vanishing set is one of its largest connected subsets, i.e., a connected subset contained in no other such subset. The "quasi-component" of A belonging to the point p consists of the points belonging to the same summand as p in every decomposition of A as the sum of two sets closed in A and having the null-set as section. The quasi-components may differ from the components. After a series of theorems on connectivity, the chapter is devoted to density, and to the application of some of the results to sets of real numbers.

In Chapter VIII (pages 260–358) special topological spaces are considered. A stride towards ordinary space is made by the successive introduction of the denumerability postulates:

(E) The set of neighborhoods of x is denumerable for every x.

(F) The totality of all neighborhoods is denumerable.

With the aid of (E), it follows, for example, that every convergent set (= set having a limit) is countable; and that if x is a β-point of a set A, there is a convergent subset of A with x as limit. With (F) the γ-points begin to play an important rôle. If a set has no γ-points belonging to it, it is countable. The following equations hold ($A_{\alpha\alpha}$ = set of α-points of A_α, etc.): $A_{\alpha\alpha} = A_\alpha$, $A_{\alpha\beta} = A_\beta$, $A_{\alpha\gamma} = A_{\beta\gamma}$; $A_{\beta\alpha} = A_\beta$; $A_{\gamma\alpha} = A_\gamma$, $A_{\gamma\beta} = A_\gamma$, $A_{\gamma\gamma} = A_\gamma$. A set of regions no pair of which have common points is countable. There are in all $\aleph$ regions and $\aleph$ closed sets ($\aleph$ = cardinal number of the continuum). The sum of any number of regions is the sum of a countable number (at most) of them.

By means of this result, Borel's theorem may be extended and its complete converse given.

The connection of point set theory with the transfinite ordinals lies in such theorems as the following (proved with the aid of postulate (F)): An ascending normally ordered set $\{S_\xi\}$—i.e., $S_\rho \supset S_\sigma$ for $\rho > \sigma$—of different regions S_ξ is at most countable. Similarly for a descending set of regions and for relative regions, and likewise for closed and relatively closed sets. It is at this juncture that Hausdorff introduces his generalization of "reducible" sets. It turns out that a Hausdorff reducible set is representable as the sum of differences of descending normally ordered closed sets, and conversely.

Further developments refer to metric spaces: "We believe the time has come when a continuation of the neighborhood theory would be accompanied with a loss of simplicity." A metric space satisfies the following postulates, where $\overline{xy}$ denotes the "distance"—Fréchet's écart—from x to y:

(α) (Postulate of symmetry) $\overline{yx} = \overline{xy}$.

(β) (Postulate of coincidence) $\overline{xy} = 0$ when and only when $x = y$.

(γ) (Triangle postulate) $\overline{xy} + \overline{yz} \geqq \overline{xz}$.

Space will not permit a discussion of the rich content of the rest of the chapter. We mention: distances between sets, connectivity properties of metrical spaces including ρ-connectivity, properties related to the various limits of a sequence of sets, Borel sets, conditions for compactness and "complete" (vollständige) spaces. After certain general theorems concerned chiefly with connectivity in euclidean n-space, the chapter turns to the euclidean plane—see quotation in early portion of this review—and after a succession of thirteen carefully graded theorems, culminates in a proof—modeled after that of Brouwer—of the Jordan theorem.

The ninth chapter (pages 358–399) deals with representations or functions. If the original set A and its image B (= totality of elements $f(a)$, where a ranges over A) are topological spaces, the continuity of $f(a)$ is equivalent to the condition that every relative region of B is the image of a relative region of A. If f is continuous and A connected, then B is connected. From this it follows, in particular, that a continuous, real function defined in a connected set—for example, in a linear interval—takes every value between any

two of its values. Among other things, the chapter considers uniform continuity for metrical spaces, continuous curves including the space-filling curves, the character of the points of continuity of a discontinuous function, sequences of functions and the generalizations of theorems of Arzelà and Baire. There are novel results on the classification of functions and the set of convergence points of a sequence of functions.

The tenth and final chapter (pages 399–448) treats of Peano-Jordan content and Borel-Lebesgue measure; both theories are developed in a new, elegant and appreciably similar fashion. There are applications to decimal and to continued fractions. After the deduction of the important properties of Lebesgue integrals, the chapter closes with the proof that a function of limited variation possesses a derivative except in a set of measure zero.

The appendix (pages 449–473) gives references to the literature and contains numerous discussions of substantial content and interest.

Misprints are few in number; and errors, invariably of a minor character. The mention of most of them would be regarded as hypercritical, were it not for the high standards of the author. The careful examination of the reviewer has brought to light only the following: page 11, line 15, "Elemente" instead of "Punkte"; page 28, line 27, "Relationen" instead of "Gleichungen"; page 58, line 14 and page 105, line 20, "Denn" instead of "Dann"; page 272, G_{i_1} is not printed clearly; page 366, line 6, insert "Kap. VIII" after "nach"; page 442, line 5, insert Σ before $\mu_i\delta_i$. On pages 85 and 291, in the footnotes, occur the equations $m = \{m\}$, $x = \{x\}$; the author clearly intimates their objectionability, but it would be better to bar altogether such illogical statements. On page 106, the author speaks of numbers greater than W, when he means numbers greater than numbers of W. On page 229, he forgets to discuss A_k at the end of the section. On page 276, the restriction that λ shall be a limiting number does not enter the proof. This is an exception to prove the rule that in the Grundzüge every word counts. On page 444, the statement that X is normally ordered, although later cleared up, contains at first an ambiguity; for X may be normally ordered without being ordered according to magnitude. On page 470, $\psi^{-m_1}\delta\psi^{m_1}$ is not necessarily an α, but may be a γ; it is an easy matter, however, to fill out the gap.

As for more important criticism, one may quarrel with the author for his abstract style, for his euclidean manner of grading the proofs, so that no difficulties remain and none but mild climaxes are reached, for his finish that may excite admiration but hardly activity on the reader's part. One may crave for a book that is built like a drama around a single idea —a more sketchy book, leaving more to the reader's imagination, a book with a less diversified and more emphatic message. But such remonstrance would be like quarrelling with Beethoven for having written symphonies instead of operas. There is no such thing as *the* book. Hausdorff's Grundzüge is a treatise, and as a treatise it necessarily falls short of the summum bonum. But as a treatise it is of the first rank.

HENRY BLUMBERG.

SHORTER NOTICES.

The Casting-Counter and the Counting Board. By FRANCIS PIERREPONT BARNARD. Oxford, Clarendon Press, 1916. 358 pp. + 63 plates.

When we consider that Gerbert, the greatest mathematician living in Europe at the close of the tenth century, wrote upon the use of counters as an aid to computation; that Robert Recorde, who is often called the founder of the English school of mathematicians of the sixteenth century, did the same; and that nearly all computation in Europe before the year 1500 (in Italy before c. 1200) was performed by the aid of some type of abacus, we may well infer that the "casting-counter," as Professor Barnard calls it, has played an important rôle in the history of calculation. Indeed, our very word "calculate" is, it need hardly be said, due to this very fact, the word "calculus" meaning a pebble, calculi being used in numerical work in the classical period of the Greek and Roman civilizations.

When we also consider the fact that it was the bamboo rods, used by the early Chinese algebraists to express coefficients, that suggested to the Japanese the *sangi* which were used for the same purpose, and also suggested the idea of determinants which their scholars developed in the seventeenth

century, anticipating the expansion of these forms by Leibniz, the counter takes on a dignity that might hardly have been anticipated. And finally when we consider that it was the humble abacus that led to such devices as the suan pan, the soroban, the tschotü, and the choreb now used in, numerically speaking, the greater part of the civilized world; that led Pascal to invent and Leibniz to work upon the modern calculating machine; and that finds frequent place in the literature of Chaucer, Shakespeare, and other makers of our language, it may well occur to the mathematician to look with interest upon the treatise under review, which Professor Barnard has published in such sumptuous form.

The counters that have come down to us date from the thirteenth to the eighteenth century, the earlier pieces having been melted up for the metal or lost because of the very fact that they were too common to be held in much esteem. There are, of course, many disks of bone, baked earth, or metal that have come down to us from ancient times and which may have been used for purposes of calculation by the calculones, the calculatores, or the numerarii; but since they bear no inscriptions, we are uncertain of their use and they may have been merely pieces employed in playing such ancient games as backgammon or checkers. From the thirteenth century on for five hundred years, however, specimens were preserved, and these still remain to tell the story, decade by decade, of the use of these devices, of their change in form, of the Rechenmeisters for whom they were struck, and of the slow decay of the abacus as the power of the eastern numerals came to be the better understood. To-day, about all that we have in common use to remind us of the ancient and medieval counter (projectilis, jetton, augrim stone, calculus, abaculus, Worpghelt, Legpennig, Rechenpfennig, méreau à compte, tessera ad computandum, and the like) of our ancestors—is the poker chip, the wire with its disks above the billiard table, and the string of beads used in various religious ceremonials throughout the Buddhist, Mohammedan, and Christian world.

In the work under review Professor Barnard has given us the benefit not only of knowing about his collection of some 7,000 jettons and about some 40,000 other specimens which he has examined, but of benefitting by his rare scholarship in all that relates to medieval history. As to the former, he

has reproduced by photographic process a hundred and twenty-six of the rarest counters in his collection, generally both in obverse and reverse, and has given scientific descriptions of a large number of others. As to the opportunity that he has given the reader of having some share in his scholarship—he has shown the painstaking care in research into historical details, and the interesting style of presenting facts that characterize his other writings upon the period to which he has devoted a life of study. The fact that the bibliography of works consulted contains more than six hundred items is proof of the care with which he has considered the literature relating to the subject.

The work consists of three parts. Part I relates to the history of the casting-counter and to descriptions of the specimens found in England, Italy, France, the Low Countries, Germany, and Portugal. Part II contains the only worthy description that we have of the counting boards and counting cloths used in the medieval and Renaissance periods, and is admirably illustrated by numerous plates. Part III sets forth the methods of casting with jettons, a subject upon which we have plenty of information from such sources as Robert Recorde (c. 1542), John Awdeley (1574), Nicholas Cusa (1514), Martinus Siliceus (1526), Köbel (1514), and various other writers of the sixteenth century, upon whose works the author has freely drawn.

There are two elaborate indexes, one of legends and inscriptions on the counters and the other of a general nature—and nothing is more conducive to the comfort of a student who has occasion to consult a work of this nature than a good index.

It may be said of the work as a whole that it represents the most elaborate study that we have upon any of the minor features of this kind in the history of mathematics, and that it deserves a place in every college and university library and on the shelves of every one who is working in the special field of the history of computation. DAVID EUGENE SMITH.

Tables des Nombres Premiers, et de la Décomposition des Nombres de 1 *à* 100,000. By G. INGHIRAMI, reviewed and corrected by Dr. PROMPT. Gauthier-Villars et Cie., 1919. xi + 35 pp.

THIS little factor table gives the smallest divisor of all

numbers less than 100,000 which are prime to 2 and 5. The inclusion of the smallest factor 3 adds little to the value of the table and much to the bulk. It is decidedly inferior to Burckhardt's table in convenience of arrangement, and is approximately twice as bulky as a table which omits multiples of 2, 3, 5 and 7. It is difficult to see what end is served by the republication of such a table at the present time.

The book also contains a table of "Tessaréen" numbers, which seem to be prime numbers of the form $a^2 - a - 1$. The connection between such numbers and the numbers representable by the binary quadratic form $x^2 - xy - y^2$ is not indicated. A list of such numbers with the corresponding values of a is given, the list extending as far as the prime 19991. What particular use is to be made of this table is not indicated. The "preliminary explanation" avoids giving any demonstration of the properties of "Tessaréen" numbers, on the ground that such a demonstration would not teach anything to one who was already familiar with the theory of numbers, and would not be understood by one who was not.

The author of the preliminary explanation—name not signed, but perhaps Lebon—deplores the neglect of the theory of numbers, which neglect he attributes to the vicious methods and barbarous terminology of German mathematicians. Gauss especially comes in for a thorough castigation for his bizarre and incoherent formulas!

Five pages of the introduction are devoted to a biography of Inghirami by Giovanni Giovannozzi. Inghirami's most important work seems to have been done in astronomy and in geodesy. The factor tables here republished appeared for the first time in 1832 at the end of a volume on Elementi di Matematiche. He evidently did not know of Burckhardt's tables published some twenty years previously.

D. N. LEHMER.

Lectures of the Theory of Plane Curves. By SURENDRAMOHAN GANGULI, M.Sc., Lecturer in Pure Mathematics, University of Calcutta. Part I, x + 140 pp.; Part II, xiii + 350 pp., and 13 pages of figures. Published by the University of Calcutta, 1919.

THESE lectures were delivered to postgraduate students and comprise in a fairly satisfactory manner most of the topics usually presented in an elementary course on plane curves.

The first part is concerned with the general theory, the second with cubics and quartics. In teaching the subject, Ganguli had constant recourse to the classic treatises of Salmon and Clebsch, and the works of Basset, Scott, and others, and laid particular stress on Sylvester's theory of residuation. But instead of using this theory for the systematic study of the geometry of point groups on curves, Ganguli merely makes occasional applications of it. The definition of a curve based upon the representation of a function, in §§ 10–11, is confusing and inadequate. In the first place there is no proper postulation of the space in which curves are to be defined. Metric and projective concepts are indiscriminately mixed up, so that, as a consequence, a proper formulation of the systems of coordinates is impossible. From the present prevailing critical point of view this is one of the weakest points of Ganguli's lectures. What, for example, do the following statements mean, when the graph of a function is not defined? "In the modern theory of functions, it is held that a function can be completely defined by means of a graph arbitrarily drawn in the finite and continuous domain of the independent variable." Again, notice the erroneous declaration, "The modern theory of functions says that the equation $F(x, y) = 0$ cannot in general represent a curve, it can do so if y can be expressed as a regular (or rational) function $f(x)$ of x, i.e., if $f(x)$ is a continuous, finite and differentiable and separately monotonous function. It is only by a combination of these conditions that $y = f(x)$ can represent a curve." From this it is evident that the author still relies on the hazy notions about a curve as they were held at the time of Euler. Not the slightest attention is given to complex domains, or to what kind of an equation $F(x, y) = 0$ is.

The principle of duality and the corresponding use of line coordinates are brought in incidentally and without an effort at systematic representation. Using the designation "reciprocal curves" suggests the idea of the special duality involved in polar reciprocity. Moreover the lack of a proper projective space makes an adequate treatment of the behavior of curves at infinity impossible.

The typography of both volumes could be considerably improved in an eventual future edition. Much more attention should also be given to the reading of proofs.

But aside from the defects mentioned above the beginner

may learn a good deal about the properties of algebraic curves, so that in this respect the publication of a new English treatise on curves is not without value, and deserves commendation.

ARNOLD EMCH.

Nouvelles Méthodes de Résolution des Equations du 3e Degré. By le VTE. DE GALEMBERT. Paris, Vuibert, 1919. 22 pp.

This pamphlet gives a method for rapid numerical calculation of the real roots of the cubic equation

$$x^3 + Ax^2 + Bx + C = 0,$$

new as far as I am aware, for the case in which there are three real roots. The equation is reduced to the form $z^3 - 3z = y$; the latter equation has three real roots if $|y| \leqq 2$. A table of corresponding values of z and y is computed once for all, by means of which the values of z may be found accurately to two decimals, whenever y is known to six places. An additional table gives z and y in terms of x, A, B and C, for the different combinations of signs which the coefficients may have in the general equation. The actual calculation of the roots is very much simplified in this way.

A. DRESDEN.

The Integral Calculus. By JAMES BALLANTYNE. Boston, Published by the author, 1919. 41 pp.

THE subtitle of this small book is sufficiently descriptive of its scope. It reads, "On the integration of the powers of transcendental functions, new methods and theorems, calculation of Bernouillian numbers, rectification of the logarithmic curve, integration of logarithmic binomials, etc."

The author has several new series expansions of transcendental functions; but does not burden his tale with arguments as to the rigor of his methods or the general validity of his formal results. The book gives the impression of having been written for the fun of it, by a very ingenious gentleman, who was having a fine time giving free rein to his analytical processes and going gladly wherever those steeds—dangerous if unchecked—might lead him.

The style of the text may be indicated by such expressions of olden time flavor as, "integrals of even powers of sin xdx to radius 1"; "the value of C is the area of the full quadrant

of the curvilinear"; "the two curves $y = 1/\sin^m x$ and $y = 1/\cos^m x$ having their origin at opposite ends of the axis of $x \cdots$"; "when m is an odd negative, there will always appear in the series one irrational term, namely $\pm \frac{1}{0} \tan^0 x$"; "that is, the logarithm of an infinite number, multiplied by zero = nothing."

In section 1 various reduction formulas for integrals of $\sin^m x$, $\cos^m x$, etc., are developed and also formulas expressing those integrals in series of trigonometric functions, e.g.: "The integral of $\sin^m x dx$ for any value of m is

$$A(1 - \cos x) - \frac{B}{3}(1 - \cos^3 x) + \frac{C}{5}(1 - \cos^5 x) - \frac{D}{7}(1 - \cos^7 x) + \cdots,$$

where $A, B, C, D, \cdots$ are the successive terms in the development of the binomial $(1 + 1)^{(m-1)/2}$." These finite and series developments are equated in section 2 to derive some well known and new formulas. For example, if we let $x = \frac{1}{2}\pi$ in the above series, for $m = 0$, and equate it to $\int \sin^0 x dx = x$, we obtain

$$1 + \frac{1}{2 \cdot 3} + \frac{3}{2 \cdot 4 \cdot 5} + \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots = \pi/2.$$

Again by equating $\int \cos^0 x dx = x$ to

$$\int \cos^m x dx = A \sin x - \frac{B}{3}\sin^3 x + \frac{C}{5}\sin^5 x \cdots,$$

we have

$$x = \sin x + \frac{1}{2 \cdot 3}\sin^3 x + \frac{3}{2 \cdot 4 \cdot 5}\sin^5 x \cdots$$

"an inversion of the series given by Newton for the sine in terms of the arc."

From the series expansion, for m odd,

$$\int_0^{\pi/2} \sin^m x dx = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdots \frac{m-1}{m}$$

and from the finite expansion, for m even,

$$\int_0^{\pi/2} \sin^m x dx = \frac{\pi}{2}\left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{m-1}{m}\right)$$

the author concludes, "But if m be considered infinite, the distinction between odd and even values of m vanishes and these two expressions will be equal to each other." It follows that

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9}\cdots,$$

the expression given by Wallis.

Section 3 takes up the series for expressing the tangent in terms of the arc and its relation to the Bernoullian numbers. From $\int \tan x dx = -\log\cos x$, the author gets

$$\tan x = x + \frac{2}{2\cdot 3}x^3 + \frac{16}{2\cdot 3\cdot 4\cdot 5}x^5 + \frac{272}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + \cdots$$

and works out a scheme for finding the numerators N_a of the terms of this expansion, namely,

$$N_a = \pm 1 \mp aN_1 \pm \frac{a(a-1)(a-2)}{2\cdot 3}N_3 \mp \frac{a(a-1)(a-2)(a-3)(a-4)}{2\cdot 3\cdot 4\cdot 5}N_5 \pm \cdots$$

to the term N_{a-2}, where a is the index of the power of x whose coefficient is required and using the upper or lower sign according as $\frac{1}{2}(n-1)$ is even or odd. The author concludes this section with a practical method for the calculation of the Bernoullian numbers. The derivation of the formula is by the theory of differences, but it gives the numbers with more facility than the usual formulas. This formula is

$$1 = \frac{1}{P+1} + \frac{1}{2} + \frac{P}{2}B_1 - \frac{P(P-1)(P-2)}{2\cdot 3\cdot 4}B_2 + \frac{P(P-1)(P-2)(P-3)(P-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}B_3 - \cdots,$$

a series terminating at $B_{P/2}$. By assuming positive, even values for P the numbers B_i may be read off.

In section 4 integrals of powers of transcendental functions in terms of these functions are given. The procedure is as follows: Let $y = \varphi(x)$. Express dy/dx in terms of y and

integrate $y^m/(dy/dx)$ with respect to y (usually in series form) "as an independent variable and as an algebraic function." The result will be the integral of $\varphi^m(x)dx$ expressed in terms of the function itself, whether it be algebraic or transcendental. The formulas derived at this point are of immediate applicability to the quadrature and rectification of, for example, such troublesome curves as the logarithm.

The final section on "the integration of the logarithm of binomials and other complex quantities" is less satisfactory and of less interest than the earlier sections and is made up of various odds and ends.

E. GORDON BILL.

NOTES.

AT the meeting of the National Academy of Sciences held at Princeton University November 16–17, 1920, the following mathematical papers were read: By E. B. WILSON: "Equipartition of energy"; by EDWARD KASNER: "Einstein gravitational fields: orbits and light rays"; by J. W. ALEXANDER: "Knots and Riemann spaces"; by PHILIP FRANKLIN: "The map coloring problem."

THE July number (volume 42, number 3) of the *American Journal of Mathematics* contains the following papers: "The failure of the Clifford chain," by W. B. CARVER; "On the representations of numbers as sums of 3, 5, 7, 9, 11 and 13 squares," by E. T. BELL; "On a certain class of rational ruled surfaces," by ARNOLD EMCH.

THE opening (September) number of volume 22 of the *Annals of Mathematics* contains: "On multiform functions defined by differential equations of the first order," by PIERRE BOUTROUX; "Hermitian metrics," by J. L. COOLIDGE; "On the expansion of certain analytic functions in series," by R. D. CARMICHAEL; "Notes on the cyclic quadrilateral," by F. V. MORLEY: "Note on the preceding paper," by F. MORLEY; "Qualitative properties of the ballistic trajectory," by T. H. GRONWALL.

IT is announced that because of the greatly increased cost of printing the publication of the *Journal de Mathématiques pures et appliquées*, founded by Liouville in 1836, will be discontinued with the current volume (series 8, volume 1) unless a considerable number of new subscriptions are secured at once. If any American mathematicians or libraries have suspended their subscriptions because of war conditions, they should send in their orders for renewal as soon as possible.

THE following six American doctorates in mathematics should be added to the list published in the November BULLETIN, making a total of twenty-one conferred in 1919–20: H. R. BRAHANA, Princeton: "Curves on surfaces"; H. J. ETTLINGER, Harvard: "I. Existence theorems for the general real self-adjoint linear system of the second order. II. Oscillation theorems for the real self-adjoint linear system of the second order"; E. S. HAMMOND, Princeton: "Periodic conjugate nets of curves"; G. M. ROBISON, Cornell: "Divergent double sequences and series"; BIRD M. TURNER, Bryn Mawr: "Plane cubics with a given quadrangle of inflexions"; W. L. G. WILLIAMS, Chicago: "Fundamental systems of formal modular seminvariants of the binary cubic."

PROFESSOR J. H. TANNER, of Cornell University, and Mrs. TANNER have given to the trustees of that institution fifty thousand dollars to establish a mathematical institute under the following stipulations. The money is to be allowed to accumulate without diversion for seventy-five years. At the end of that time one professor shall be appointed, whose duty it shall be to begin the formulation of plans for the proposed institute. At the end of each of the four succeeding periods of five years one or more additional professorships shall be established, the incumbents to collaborate in the same plans. The stipends of these professors shall be paid from the fund, but no other demands shall be made upon it until one hundred years from the date of the deed of gift (June, 1920), from which time on the income of the entire sum shall be devoted to the maintenance of the institute; half of the expenditure of each year is to be applied to research in the mathematical sciences.

IT is announced that a new edition of Professor H. S. CARSLAW's book on Fourier's series and integrals, published

in 1906 by Macmillan and for some time out of print, is now in press. The book has been completely rewritten and will consist of two volumes, the first dealing with infinite series and integrals, with special reference to Fourier's series and integrals, and the second with the conduction of heat. It is expected that the first volume will appear early in 1921, and the second in the course of that year.

APPLICATIONS for appointment to the two Benjamin Peirce instructorships at Harvard University for the year 1921–22 should reach Professor W. F. OSGOOD on or before February 1, 1921, accompanied by the necessary papers. Each instructor is now required to give two elementary courses and one other course ordinarily of an advanced character, but has no administrative duties. The salary has been increased to $2000.

THE Paris academy of sciences has awarded its Poncelet prize to Professor ELIE CARTAN, of the University of Paris, and its Francœur prize to Professor RENÉ BAIRE, of the University of Dijon.

THE Italian society of sciences (the XL) has awarded its gold medal for 1920 to Professor A. SIGNORINI, of the University of Palermo, for his papers published during the last five years.

THE new Greek mathematical society, founded in 1918, now numbers about one hundred members.

PROFESSOR ALBERT EINSTEIN, of the University of Berlin, has accepted the chair of science at the University of Leyden. He will divide his time between the two institutions.

PROFESSOR M. DISTELI has been appointed professor of geometry at the University of Zurich.

PROFESSOR F. GONSETH has been appointed professor of mathematics at the University of Bern.

PROFESSOR MICHEL PLANCHEREL, of the University of Freibourg, has been appointed to a professorship at the Zurich technical school, as successor to the late Professor A. HURWITZ.

At the University of Cagliari, Dr. A. COMESSATTI, of the University of Padua, and Dr. M. PICONE, of the University of Catania, have been appointed to associate professorships of mathematics.

DR. A. PALATINI, of the University of Padua, has been appointed associate professor of rational mechanics at the University of Messina.

PROFESSOR G. ARMELLINI, of the University of Padua, has been transferred to a full professorship of celestial mechanics at the University of Pisa.

PROFESSOR C. J. DE LA VALLÉE POUSSIN has been elected president of the recently formed Fédération belge des sociétés des sciences mathématiques, physiques, naturelles, médicales et appliquées.

PROFESSOR CHARLES RIQUIER, of the University of Caen, has been elected correspondent of the Paris academy of sciences in the section of geometry, as successor to the late Professor H. G. ZEUTHEN, and Professor L. T. QUEVEDE, of the University of Madrid, has been elected correspondent in the section of mechanics, as successor to the late Professor BOULVIN.

IN the faculty of sciences of the University of Paris, the following changes have been made. The professorship of the theory of functions has been changed to a professorship of theoretical and celestial physics. Dr. EMILE BOREL, professor of the theory of functions, has been appointed professor of the calculus of probabilities and mathematical physics, as successor to Professor V. J. BOUSSINESQ, who has retired. Dr. PAUL PAINLEVÉ, professor of rational mechanics, has been appointed professor of analytical and celestial mechanics, as successor to Professor PAUL APPELL; Professor ELIE CARTAN succeeds Professor PAINLEVÉ in the chair of rational mechanics, and Professor ERNEST VESSIOT, recently appointed assistant director of the Ecole normale supérieure, succeeds Professor CARTAN in the chair of the differential calculus. Dr. J. DRACH has been appointed professor of general mathematics, and Dr. PAUL MONTEL maître de conférences in mathematics.

PROFESSOR PIERRE BOUTROUX has been appointed professor of the history of sciences at the Collège de France.

PROFESSOR GABRIEL KOENIGS has been elected member of the Conseil supérieure de l'instruction publique, as successor to Professor APPELL.

PROFESSOR M. G. HUMBERT, of the Ecole polytechnique, has retired from active teaching, with the title of honorary professor.

PROFESSOR E. FABRY, of the University of Montpellier, has been appointed professor of integral and differential calculus at the University of Aix-Marseille.

DR. RENÉ GARNIER, of the University of Poitiers, has been promoted to a professorship of theoretical and applied mechanics.

MR. A. E. JOLIFFE, of Corpus Christi College, Oxford, has been appointed professor of mathematics at the Royal Holloway College, London.

LT. COL. A. R. RICHARDSON has been appointed professor of mathematics at University College, Swansea.

PROFESSOR L. E. DICKSON was appointed delegate of the National academy of sciences to attend the conference called by the Royal society of London in September to consider the future of the International catalogue of scientific literature.

AT Colgate University, associate professor A. W. SMITH has been made full professor and head of the department of mathematics as successor to Professor J. M. TAYLOR, who has retired. Professor T. R. AUDE, of the Carnegie Institute of Technology, has been appointed associate professor of mathematics.

PROFESSOR J. K. LAMOND, of Pennsylvania College, Gettysburg, has resigned to accept a position in the engineering department of the Bell Telephone Company at Philadelphia.

At the U. S. Naval Academy, assistant professor G. R. Clements has been promoted to an associate professorship of mathematics, and Dr. L. S. Dederick and Dr. L. T. Wilson to assistant professorships.

At the University of Wyoming, Professor C. B. Ridgaway has retired, after twenty-four years of service, and Professor C. E. Stromquist succeeds him as head of the department of mathematics. Mr. A. R. Fehn, of the Kansas Agricultural College, has been appointed associate professor of mathematics.

Associate professor H. H. Conwell, of the University of Idaho, has resigned to accept a similar position in the department of mathematics at Beloit College.

Dr. Lennie P. Copeland and Dr. Mary F. Curtis have been promoted to assistant professorships of mathematics at Wellesley College.

At the University of North Carolina, Professor William Cain has retired from active teaching, and Professor Archibald Henderson succeeds him as head of the department of mathematics; assistant professors A. W. Hobbs and J. W. Lasley, Jr., have been promoted to associate professorships.

In the department of mathematics at Purdue University, the following changes have been made: assistant professor W. A. Zehring has been promoted to an associate professorship; Mr. C. T. Hazard, Mr. C. K. Robbins, and Dr. G. H. Graves have been promoted to assistant professorships; Dr. E. M. Berry, Mr. W. H. Frazier, Mr. W. R. Hardman, and Mr. J. A. Needy have been appointed instructors.

At the University of Saskatchewan, Dr. I. A. Barnett, of the University of Illinois, has been appointed assistant professor of mathematics. Dean G. H. Ling has been granted leave of absence for the current year, and is at present in England.

Professor I. L. Miller, of Carthage College, has been appointed associate professor of mathematics at the South Dakota State College.

ASSISTANT professor HILLEL HALPERIN, of the University of Arkansas, has been appointed to an associate professorship at the Texas Agricultural and Mechanical College.

PROFESSOR J. B. FAUGHT, of the State normal college at Kent, Ohio, has been appointed professor of mathematics at Yankton College.

PROFESSOR P. H. GRAHAM, of Agnes Scott College, has been appointed instructor in mathematics at New York University.

MR. R. M. MATHEWS, of the University of Minnesota, has been appointed assistant professor of mathematics at Wesleyan University.

MR. B. D. ROBERTS has been appointed professor of mathematics at Parsons College, Fairfield, Iowa.

MR. C. D. EHRMAN, of the University of Wisconsin, has been appointed assistant professor of mathematics at Richmond College.

MISS FRANCES B. HATCHER has been appointed associate professor of mathematics at Westhampton College, Richmond, Va.

DR. H. F. MACNEISH has been promoted to an assistant professorship of mathematics at the College of the City of New York.

DR. C. L. E. WOLFE, of the Junior College at Santa Rosa, has been appointed instructor in mathematics at the California Institute of Technology, Pasadena.

PROFESSOR H. W. STAGER, of Fresno Junior College, has been appointed instructor in mathematics at the University of Washington.

AT Harvard University the following instructors in mathematics have been appointed for the current academic year: professor C. A. GARABEDIAN, of New Hampshire College; Mr. R. E. LANGER, Mr. E. L. MACKIE, and Mr. H. LEVY, of

Harvard University; Mr. A. J. Cook, of the University of Alberta.

Mrs. M. I. Logsdon has been appointed associate in mathematics at the University of Chicago.

Miss Mary Ball has been appointed instructor in mathematics at Northwestern University.

Dr. Tobias Dantzig, of Johns Hopkins University, has resigned to undertake mathematical research for an engineering company in New York City.

Mr. T. H. Johnson, Mr. A. S. Adams, and Mr. W. L. Lucas have been appointed instructors in mathematics at the University of Maine.

Mr. J. B. Rosenbach, of the University of Illinois, has been appointed instructor in mathematics at the Carnegie Institute of Technology.

Miss Jessie M. Short, formerly of Carleton College, has been appointed instructor in mathematics at Reed College.

Professor John Perry, of the Royal College of Science, London, died August 4, 1920, at the age of seventy years.

Professor F. A. Tarleton, of Trinity College, Dublin, died June 20, 1920, at the age of seventy-nine years.

The death is reported, in August, 1920, of assistant professor H. D. Frary, of the University of Wisconsin.

Professor Samuel Hanaway, who retired in 1916 on account of ill health from the department of mathematics of the College of the City of New York, died recently at the age of sixty-six years.

Professor E. W. Stanton, of the teaching staff of Iowa State College since 1870, died September 12, 1920, at the age of seventy years.

Professor Maelynette Aldrich, of Martha Washington College, died February 22, 1920.

BOOK CATALOGUES:—Librairie scientifique Emile Blanchard, Paris, Catalogue of works on mathematics, astronomy, meteorology, and navigation, 1200 titles.—Galloway and Porter, Cambridge, List of 116 titles in mathematics and physics.—Gauthier-Villars, Paris, Bulletin trimestriel, 2e trimestre, 1920.—Macmillan, New York, Catalogue of books on mathematics, astronomy, and navigation, 1920–1921.—Bücher, Musikalien, Lehrmittel, Kunstblätter, ein Verzeichnis herausgegeben von 65 deutschen Verlegern, Leipzig, 1920.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ALEXANDER (S.). Space, time, and deity. 2 volumes. London, Macmillan, 1920. 16 + 347 + 13 + 437 pp. 36s.

BADOUREAU (A.). Causeries philosophiques. Paris, Gauthier-Villars, 1920. 8vo. 20 + 226 pp. Fr. 12.00

BOREL (E.). See GIRAUD (G.).

BRODETSKY (S.). A first course in nomography. London, Bell, 1920. Demy 8vo. 10s.

BROGGI (U.). Análisis matemático. Volumen 1: Las nociones fundamentales. La Plata, Facultad de ciencias físicas, matemáticas y astronómicas, 1919. Royal 8vo. 152 pp. 6 pes.

BROUWER (L. E. J.). Wiskunde, waarheid, werkelijkheid. Groningen, Noordhoff, 1919. 29 pp.

GIRAUD (G.). Leçons sur les fonctions automorphes. Fonctions automorphes de n variables. Fonctions de Poincaré. (Collection de monographies sur la théorie des fonctions publiée sous la direction de M. Emile Borel.) Paris, Gauthier-Villars, 1920. 8vo. 4 + 128 pp. Fr. 13.00

HALSTED (G. B.). See SACCHERI (G.).

HARTINGER (H.). Ueber Komplexe, die sich erzeugen durch Kongruenzen 1ter Ordnung, 2ter Klasse, deren Brennlinien auf einer Kegelfläche 2ten Grades liegen. (Dissertation.) Technische Hochschule, München, 1916.

MANGOLDT (H. VON). Einführung in die höhere Mathematik für Studierende und zum Selbstunterricht. 2ter Band. 2te Auflage. Leipzig, 1919. M. 26.00

SACCHERI (G.). Euclides vindicatus; translated with introduction and notes by G. B. Halsted in a Latin-English edition. Chicago, Open Court, 1920. 8vo. 280 pp. $2.00

SCHÜRER (F.). Integraldifferenzen- und Differentialdifferenzengleichungen. Leipzig, Teubner, 1919. (Preisschriften, gekrönt und herausgegeben von der Fürstlich Jablonowskischen Gesellschaft.) M. 7.20

WALSEMANN (H.). Zahlenlehre für Grundschulen. Eine Anleitung zur Entwicklung des Zahlensinnes und Zahlenverstandes mit Benutzung handlicher Zahlenfelder. Schleswig, Selbstverlag, 1919. M. 2.80

WIELEITNER (H.). Algebraische Kurven. Neue Bearbeitung. 1ter Teil: Gestaltliche Verhältnisse. 2ter Teil: Allgemeine Eigenschaften. Neudruck. Berlin, 1919. M. 2.10 + 2.10

II. ELEMENTARY MATHEMATICS.

BARDEY (E.). Arithmetische Aufgaben nebst Lehrbuch der Arithmetik. Neue Ausgabe nach F. Pietzker und O. Presler bearbeitet von G. Mohrmann. 6te Auflage. Leipzig, Teubner, 1919. 333 pp. Geb. M. 3.80

CUSACK (J.). The arithmetic of the decimal system. London, Macmillan, 1920. 16 + 492 pp. 6s.

DELBRIDGE (C. L.). Delbridge freight calculator, by rises of ½ c. per 100 lbs. to $1 per 100 lbs., and 10 c. per ton to $20 per ton by rises of 10 c. per ton. St. Louis, Delbridge Company, 1920. 200 pp. $3.00

DUFAYEL (H.). Course de comptabilité. Paris, Dunod 1920. 8vo. 228 pp. Fr. 19.50

DUNKLEY (W. G.). A primer of trigonometry for engineers. London, Pitman, 1920. 8 + 171 pp. 5s.

GODFREY (C.) and SIDDONS (A. W.). Practical geometry. Theoretical geometry. Cambridge, University Press, 1920. 15 + 256 + 14 + 104 pp. 7s.

MINELLI (R.). Trattato di geometria ad uso delle scuole secondarie. Roma, Ausonia (Spoleto, Panetto e Petrelli), 1920. 8vo. 188 pp. con 6 tavole. L. 4.00

MOHRMANN (G.). See BARDEY (E.).

PIÉRARD (E.). Théorie sommaire de la règle à calcul et son application à la résolution des équations du 3e degré. Paris, Dunod, 1920. 12mo. 16 pp. Fr. 1.50

PIETZKER (F.). See BARDEY (E.).

PRESLER (O.). See BARDEY (E.).

ROHRBERG (A.). Theorie und Praxis des logarithmischen Rechenschiebers. 2te, verbesserte und vermehrte Auflage. Leipzig, Teubner, 1919. M. 1.40

SIDDONS (A. W.). See GODFREY (C.).

III. APPLIED MATHEMATICS.

ANNUAIRE pour l'an 1920 publié par le Bureau des Longitudes. Avec des notices scientifiques de J. Renaud et C. Lallemand. Paris, Gauthier-Villars, 1920. 16mo. 822 pp. Fr. 6.00

COURQUIN (A.) et SERRE (G.). Cours d'aérodynamique pratique à l'usage des pilotes et mécaniciens-aviateurs. Paris, Gauthier-Villars, 1920. 152 pp. Fr. 9.00

DANA (R. T.). See HASKELL (A. C.).

DE DONDER (T.). Théorie du champ électro-magnétique de Maxwell et du champ gravifique d'Einstein. Paris, Gauthier-Villars, 1920. 8vo. 16 + 102 pp. Fr. 14.00

DELAUNEY (F. J.). Problèmes astronomiques. La distance des satellites. La voie lactée. Les taches solaires. Les essaims d'étoiles filantes. Le système de la terre. Paris, Gauthier-Villars, 1920. 8vo. 52 pp. Fr. 6.00

DREYER (G.). Elemente der Graphostatik. 5te, unveränderte Auflage. Ilmenau, 1920. M. 10.00

EDDINGTON (A. S.). Space, time and gravitation. An outline of the general relativity theory. Cambridge, University Press, 1920. 8vo. 8 + 218 pp. 15s.

GALKA (M.). Graphostatik. Nieder-Ramstadt, 1919. M. 9.60

GRÜBLER (M.). Lehrbuch der technischen Mechanik. 2ter Band: Statik der starren Körper. Berlin, 1919. M. 18.00

GUYOT (G.). Méthodes rationnelles d'usinage. Abaques pour la détermination des conditions de travail des machines-outils. Paris, Dunod, 1920. 4to. 20 pp. + 15 tableaux. Fr. 9.00

HASKELL (A. C.). How to make and use graphic charts. With an introduction by R. T. Dana. New York, Codex Book Company, 1919. 7 + 539 pp.

HAUBER (W.). Festigkeitslehre. 6ter Neudruck. Berlin, 1919. M. 1.25

HEGEMANN (E.). Die Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. (Aus Natur und Geisteswelt, Nr. 609.) Leipzig, Teubner, 1919. M. 1.75

HUBERT (H.). Précis des cours de mécanique appliquée et physique industrielle de l'Université de Liège. Paris, Dunod, 1920. 8vo. 1740 pp. Fr. 90.00

JÄGER (G.). Theoretische Physik. 1: Mechanik und Akustik. 5te verbesserte Auflage. Berlin, 1919.

LACAZE (H.). Cours de cinématique théorique à l'usage des candidats à la licence et aux écoles du gouvernement. Paris, Gauthier-Villars, 1920. 138 pp. Fr. 16.00

LALLEMAND (C.). See ANNUAIRE.

MASCART (J.). See REYNAUD (P.).

MAURER (P.). Radiotélégraphie pratique et radiotéléphonie. Paris, Dunod, 1920. 8vo. 386 pp. Fr. 21.00

MONTEL (B. L.). Termodinamica: lezioni tenute al r. Politecnico di Torino e raccolte dall'ing. E. G. Serra. Torino, ditta Antonietti, 1919. 8vo. 187 pp. L. 12.00

MUNCK (M.). Isoperimetrische Aufgaben aus der Theorie des Flugs. (Dissertation.) Göttingen, 1919. M. 0.80

OLSCHKI (L.). Geschichte der neusprachlichen wissenschaftlichen Literatur. 1ter Band: Die Literatur der Technik und der angewandten Wissenschaften vom Mittelalter bis zur Renaissance. Heidelberg, Winter, 1918. 12 + 459 pp.

PICARD (E.). Notice historique sur la vie et l'œuvre de lord Kelvin, lue dans la séance publique annuelle de l'Académie du 22 décembre 1919. Paris, Gauthier-Villars, 1920. 4to. 40 pp. Fr. 3.00

PLANCK (M.). Vorlesungen über die Theorie der Wärmestrahlung. 3te Auflage. Leipzig, 1919. M. 15.00

PULFRICH (C.). Ueber Photogrammetrie aus Luftfahrzeugen und die ihr dienenden Instrumente. Jena, 1919. M. 4.00

RENAUD (J.). See ANNUAIRE.

REYNAUD (P.). Etude sur le système solaire. Nouvelle loi de distances des planètes et des satellites. Préface de J. Mascart. Paris, Gauthier-Villars, 1919. 8vo. 82 pp.

SCHAEFER (C.). Die Prinzipe der Dynamik. Berlin, 1919. M. 8.50

SCHOTT (S.). Statistik. 2te Auflage. Leipzig, Teubner, 1920. M. 1.60

SERRA (E. G.). See MONTEL (B. L.).

SERRE (G.). See COURQUIN (A.).

SILBERSTEIN (L.). Report on the quantum theory of spectra. London, Adam Hilger, 1920. 8vo. 42 pp. 5s.

SLOANE (T. O.). Arithmetic of electricity. Revised and enlarged edition. 1920. 200 pp. $1.50

SOMMERFELD (A.). Atombau und Spektrallinien. Braunschweig, Vieweg, 1919. M. 25.00

STÜCKLEN (H.). Zur Frage der scheinbaren Gestalt des Himmelsgewölbes. (Dissertation.) Göttingen, 1919. M. 0.60

SUCKOW (F.). Die Landmessung. (Aus Natur und Geisteswelt, Nr. 608.) Leipzig, Teubner, 1919. Geb. M. 2.65

SZARVADY (G.). Théorie des enroulements des machines à courant continu. Paris, Dunod, 1919. 8vo. 137 pp. Fr. 13.50

TEICHMANN (J.). Mechanische Probleme, die auf belastete Integralgleichungen führen. (Dissertation.) Breslau, 1919.

THOMAS (W. N.). Surveying. With answers. London, E. Arnold, 1920. 8 + 536 pp. 31s. 6d.

TSCHEUSCHNER (H.). La prévision du temps sans instrument. 2e édition. Paris, Gauthier-Villars, 1919. 8vo. 18 pp. Fr. 2.50

TURRIÈRE (E.). Le problème des objectifs de longue-vue dans la dioptrique contemporaine. Exposé des recherches de M. Harting. Paris, Imprimerie du Service géographique de l'Armée, 1918. 8vo. 149 pp.

VIERENDEEL (A.). Cours de stabilité des constructions. Tome 1: Résistance des matériaux. 3e édition. Paris, Dunod, 1919. 8vo. 421 pp. Fr. 48.00

VIOLA (C.). Trattato di cristallografia. Milano, 1920. 8vo. 15 + 389 pp.

WEITBRECHT (W.). Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. 2te Auflage. Leipzig, 1919. M. 2.10